

# Exam

## Statistical Physics

Thursday January 26, 2017

9:00-12:00

**Read these instructions carefully before making the exam!**

- Write your name and student number on *every* sheet.
- *Make sure to write readable for other people than yourself. Points will NOT be given for answers in illegible writing.*
- *Language; your answers have to be in English.*
- Use a *separate* sheet of paper for each problem.
- Use of a (graphing) calculator is allowed.
- This exam consists of 4 problems.
- The weight of the problems is: Problem 1 (P1=25 pts); Problem 2 (P2=20 pts); Problem 3 (P3=20 pts); Problem 4 (P4=25 pts). Weights of the various subproblems are indicated at the beginning of each problem.
- The grade of the exam is calculated as  $(P1+P2+P3+P4+10)/10$ .
- For all problems you have to write down your arguments and the intermediate steps in your calculation, *else the answer will be considered as incomplete and points will be deducted.*

PROBLEM 1

Score:  $a+b+c+d+e+f=4+4+4+4+5+4=25$

Consider a solid that consists of a large number ( $N$ ) of atoms with spin  $\frac{1}{2}$ , each of which has a fixed position in space. Each atom has a magnetic moment  $\mu$  that can be aligned either parallel or anti-parallel with an external magnetic field  $B$ . We assume that the magnetic moment of one atom interacts only very weakly with those around it. The solid is in equilibrium at temperature  $T$ .

The energy levels of a single atom are:

$$\begin{aligned}\varepsilon_1 &= -\mu B: && \text{if the spin is parallel to the magnetic field } B \\ \varepsilon_2 &= \mu B: && \text{if the spin is antiparallel to the magnetic field } B\end{aligned}$$

We define the variable,  $x = \frac{\mu B}{kT}$ .

- Give the single-atom partition function  $z$  (express your answer in terms of  $x$ ).
- Give expressions for the probabilities  $P_1$  and  $P_2$  that the energy levels  $\varepsilon_1$  and  $\varepsilon_2$  are occupied (express your answers in terms of  $x$ ). Determine the values of  $P_1$  and  $P_2$  in the limit  $T \rightarrow 0$  and in the limit  $T \rightarrow \infty$ .
- Explain why the partition function  $Z$  for  $N$  spins is given by  $Z = z^N$  and does not need a factor  $1/N!$ .
- Use the partition function  $Z$  to show that the total energy  $E$  due to the spins of the  $N$  atoms is given by,

$$E = -NkTx \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

- Prove that the entropy  $S$  of the spins of the  $N$  atoms is given by,

$$S = Nk \left( \ln[e^x + e^{-x}] - x \frac{e^x - e^{-x}}{e^x + e^{-x}} \right)$$

- Suppose the solid is in equilibrium at a temperature  $T_1 = 10$  K. The magnetic field is then reduced from its initial value  $B_1$  to a lower value  $B_2 = 0.01B_1$  in a *reversible adiabatic* process. Calculate the temperature  $T_2$  of the system after this process.

## PROBLEM 2

Score:  $a+b+c+d+e=4+4+4+4+4=20$

A system consists of two identical non-interacting particles. The system has three single-particle states:  $\varphi_1, \varphi_2$  and  $\varphi_3$  with energies  $\varepsilon_1 = 0 < \varepsilon_2 < \varepsilon_3$ , respectively.

Use the notation  $(n_1, n_2, n_3)$  for two-particle states with  $n_i$  ( $i = 1, 2, 3$ ) the occupation number of single-particle state  $\varphi_i$  ( $i = 1, 2, 3$ ), respectively. For example:  $(1, 1, 0)$  means that one particle is in state  $\varphi_1$  and one particle is in state  $\varphi_2$ .

List all two-particle states with their energy and degeneracy in case the two particles are:

- Distinguishable classical particles
- Indistinguishable fermions
- Indistinguishable bosons
- Give the partition function for situation a), b) and c).
- Using only the two leading terms for low temperatures in the partition function find the energy as function of the temperature for situation b) and c). Compare the behaviour of the two situations in the limit  $T \rightarrow 0$ .

PROBLEM 3

Score:  $a+b+c+d+e=5+3+3+4+5=20$

A classical perfect gas of  $N$  atoms with mass  $m$  is confined to two dimensions (surface  $A = L_x L_y$ ). The number of atoms,  $N$ , is a very large number.

a) Show that the single-particle partition function  $Z_1$  of this gas is given by,

$$Z_1 = A \left( \frac{2\pi m k T}{h^2} \right)$$

HINT: The density of states for a *spinless* particle confined to a 2-dimensional enclosure with surface area  $A$  is (expressed as a function of the particle's momentum  $p$ ):

$$f(p) dp = \frac{A}{h^2} 2\pi p dp$$

b) Show that the Maxwell speed distribution of this 2D gas is,

$$P(v) dv = \frac{mv}{kT} e^{-\frac{mv^2}{2kT}} dv$$

c) Calculate the average speed of the atoms.

d) Show that the Helmholtz free energy  $F$  for this gas is given by:

$$F = -NkT \left( \ln \left( \frac{A 2\pi m k T}{N h^2} \right) + 1 \right)$$

e) Use  $F$  to derive the equation of state of this gas.

PROBLEM 4

Score:  $a+b+c+d+e = 6+5+4+5+5=25$

A gas of photons is confined to a cavity with volume  $V$ . The cavity is kept at a constant temperature  $T$ .

HINT 1: The density of states for a *spinless* particle confined to an enclosure with volume  $V$  is (expressed as a function of the particle's momentum  $p$ ):

$$f(p)dp = \frac{V}{h^3} 4\pi p^2 dp$$

HINT 2: The mean number of photons in a state with energy  $\varepsilon = \hbar\omega$  is equal to:  $\frac{1}{e^{\beta\varepsilon}-1}$

a) Show that density of states of a photon in the cavity can be written as,

$$f(\omega)d\omega = \frac{V\omega^2 d\omega}{\pi^2 c^3}$$

b) Show that the total number of photons in the cavity is given by,

$$N = b \frac{Vk^3 T^3}{\pi^2 \hbar^3 c^3}$$

where  $b$  is dimensionless constant.

c) Find the numerical value of  $b$ .

d) Show that the total energy density  $u = \frac{E}{V}$  ( $\text{J m}^{-3}$ ) in the cavity is related to the temperature  $T$  by,

$$u = aT^4 \text{ with } a = \frac{\pi^2 k^4}{15 \hbar^3 c^3}$$

The Helmholtz free energy  $F$  of the photon gas is  $F = -\frac{1}{3}aVT^4$ .

e) Prove that for the photon gas the energy  $E$  is given by,

$$E = F + TS$$

with  $S$  the entropy of the photon gas.